Geometric Means and Tensor Products

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Abstract We discover a variety of new inequalities involving the tensor product and the geometric mean of positive definite matrices. This work builds on a generalization of an inequality of Ando [2] by Feng and Tonge [4].

Keywords and Phrases. Arithmetic mean, α-power mean, geometric mean, harmonic mean, and tensor product.

1 Introduction

When M_1 and M_2 are arbitrary $n \times n$ matrices, their *arithmetic mean* is defined as

$$A(M_1, M_2) := \frac{1}{2}(M_1 + M_2),$$

and if M_1 and M_2 are positive $n \times n$ matrices, their *harmonic mean* can be defined as

$$H(M_1, M_2) := \left[\frac{1}{2}(M_1^{-1} + M_2^{-1})\right]^{-1}.$$

For this paper, if M is a positive definite $n \times n$ matrix and $\alpha \in \mathbb{R}$, then M^α will denote its unique positive α^{th} power. From the definition, one can easily verify that

$$\left(M^{-1}\right)^\alpha = (M^\alpha)^{-1} = M^{-\alpha}.$$

Ando [2] gave the definition of the geometric mean of two positive $n \times n$ matrices, M_1 and M_2, as

$$G(M_1, M_2) := M_1^{1/2} \left(M_1^{-1/2} M_2 M_1^{-1/2}\right)^{1/2} M_1^{1/2}.$$

It is clear that $G(M_1, M_2) > 0$ and when M_1 and M_2 commute, we have

$$G(M_1, M_2) = (M_1 M_2)^{1/2}.$$

Ando's definition obeys the arithmetic-geometric-harmonic mean inequality,

$$H(M_1, M_2) \leq G(M_1, M_2) \leq A(M_1, M_2).$$

If $M = (m_{ij}),\ N = (n_{ij})$ are matrices of the same size, their *Hadamard product* $M \circ N$ is the matrix of entrywise products; namely,

$$M \circ N = (m_{ij} n_{ij}).$$

Ando proved that for positive definite $n \times n$ matrices $M,\ N$ we have

$$G(M, N) \circ G(M, N) \leq M \circ N. \qquad (A_1)$$

Note that in the commutative case, this reduces to

$$(MN)^{\frac{1}{2}} \circ (MN)^{\frac{1}{2}} \leq M \circ N.$$

In addition, in the same paper Ando generalized this inequality to the case of several commuting positive definite $n \times n$ matrices to get

$$\prod_{j=1}^{m} \circ \left(\prod_{i=1}^{m} M_i \right)^{\frac{1}{m}} \leq \prod_{i=1}^{m} \circ M_i. \qquad (A_2)$$

In 1994, Sagae and Tanabe [5] extended Ando's definition of the geometric mean of two matrices to the case of several positive definite $n \times n$ matrices.

Definition 1 (Sagae and Tanabe, [5])**.** *Let $w_1, \ldots, w_k$ be positive numbers summing to 1, and let $M_1, \ldots, M_k$ be positive definite $n \times n$ matrices. Define their **weighted geometric mean** $G_w(M_1, \ldots, M_k)$ to be*

$$M_k^{\frac{1}{2}}(M_k^{-\frac{1}{2}} M_{k-1}^{\frac{1}{2}} \cdots (M_3^{-\frac{1}{2}} M_2^{\frac{1}{2}} (M_2^{-\frac{1}{2}} M_1 M_2^{-\frac{1}{2}})^{u_1} M_2^{\frac{1}{2}} M_3^{-\frac{1}{2}})^{u_2} \cdots M_{k-1}^{\frac{1}{2}} M_k^{-\frac{1}{2}})^{u_{k-1}} M_k^{\frac{1}{2}},$$

where $u_i = 1 - \left(w_{i+1} \,/\, \sum_{j=1}^{i+1} w_j \right)$ for $i = 1, \ldots, k-1$.

When $k = 2$ and $w_1 = w_2 = 1/2$ this geometric mean reduces to Ando's geometric mean. As a result of their definition, they were able to prove the following inequalities.

Theorem 1 (Sagae and Tanabe, [5])**.** *Define the weighted arithmetic mean $A_w(M_1, \ldots, M_k)$ and weighted harmonic mean $H_w(M_1, \ldots, M_k)$ to be*

$$A_w(M_1, \ldots, m_k) := w_1 M_1 + \cdots + w_k M_k$$

$$\text{and} \qquad H_w(M_1, \ldots, M_k) := \left(w_1 M_1^{-1} + \cdots + w_k M_k^{-1} \right)^{-1}.$$

Then

$$H_w(M_1, \ldots, M_k) \leq G_w(M_1, \ldots, M_k) \leq A_w(M_1, \ldots, M_k).$$

All inequalities are strict unless $M_1 = \cdots = M_k$.

Using Theorem 1, Feng and Tonge [4] were able to build a new inequality with the tensor product of matrices. Before stating the result, we need to recall the following tensor product definition. If $M = (m_{ij})$ is an $k \times l$ matrix and $N = (n_{ij})$ is an $s \times t$ matrix, then their *tensor* (or *Kronecker) product* is the $ks \times lt$ matrix

$$M \otimes N = \begin{bmatrix} m_{11}N & \cdots & m_{1l}N \\ \vdots & \ldots & \vdots \\ m_{k1}N & \cdots & m_{kl}N \end{bmatrix}.$$

The tensor product of finitely many matrices can be defined by induction. Although the proof is a multi-stage induction argument, the following is a powerful result.

Theorem 2 (Feng and Tonge, [4]). *If M_{ij} $(1 \leq i \leq m,\ 1 \leq j \leq k)$ are positive definite $n \times n$ matrices, and w_i $(1 \leq i \leq m)$ are positive scalars summing to 1, then*

$$\prod_{j=1}^{k} \otimes\, G_w(M_{1j}, \ldots, M_{mj}) \leq \sum_{i=1}^{m} w_i \prod_{j=1}^{k} \otimes M_{ij}.$$

As an application, they generalized Ando's inequality (A_1) in the following theorem.

Theorem 3 (Feng and Tonge, [4]). *Let M_i $(1 \leq i \leq m)$ be positive $n \times n$ matrices. Then*

$$G_w\left(M_{i_1}, \ldots, M_{i_m}\right) \circ G_w\left(M_{j_1}, \ldots, M_{j_m}\right) \circ \cdots \circ G_w\left(M_{k_1}, \ldots, M_{k_m}\right) \leq \prod_{i=1}^{m} \circ M_i,$$

where $\{i_1, \ldots, i_m\}, \{j_1, \ldots, j_m\}, \ldots, \{k_1, \ldots, k_m\}$ are any permutations of $\{1, 2, \ldots, m\}$.

In this paper, we will show that we can actually prove more than Theorem 2. We will generalize the relationship between the harmonic and geometric means. Furthermore, if we change the weight conditions to real numbers w_i $(1 \leq i \leq k)$ such that $w_1 > 0, w_i < 0$ $(2 \leq i \leq k)$ and $\sum_{i=1}^{k} w_i = 1$, we will prove a new series of generalized inequalities for the arithmetic, geometric and harmonic means dealing with the tensor product of matrices.

2 Main Results

Our results will make use of the the following properties of tensor products.

Lemma 1. *Let M_i be positive definite $m_i \times m_i$ matrices $(1 \le i \le k)$. Then, for any real number α,*

$$\left(\prod_{i=1}^{k} \otimes M_i\right)^{\alpha} = \prod_{i=1}^{k} \otimes M_i^{\alpha}.$$

In particular,

$$\left(\prod_{i=1}^{k} \otimes M_i\right)^{-1} = \prod_{i=1}^{k} \otimes M_1^{-1}$$

which can be found in Feng and Tonge [4].

Lemma 2 (Feng and Tonge, [4], Theorem 3)**.** *Let M_{ij} $(1 \le i \le m, 1 \le j \le k)$ be positive $n \times n$ matrices, and let w_i $(1 \le i \le m)$ be positive scalars summing to 1. Then*

$$G_w\left(\prod_{j=1}^{k} \otimes M_{1j}, \ldots, \prod_{j=1}^{k} \otimes M_{mj}\right) = \prod_{j=1}^{k} \otimes G_w(M_{1j}, \ldots M_{mj}).$$

If α is any real number, we can define the *α-power mean* (see Ando [3]) as

$$G^{(\alpha)}(M, N) := N^{\frac{1}{2}}\left(N^{-\frac{1}{2}} M N^{-\frac{1}{2}}\right)^{\alpha} N^{\frac{1}{2}}.$$

Using induction, for $k \ge 2$, we define the *general α-power mean* as

$$G^{(\alpha_1,\ldots,\alpha_k)}(M_1, \ldots, M_{k+1}) := M_{k+1}^{\frac{1}{2}}\left(M_{k+1}^{-\frac{1}{2}} G^{(\alpha_1,\ldots,\alpha_{k-1})}(M_1, \ldots, M_k) M_{k+1}^{-\frac{1}{2}}\right)^{\alpha_k} M_{k+1}^{\frac{1}{2}},$$

where $\alpha_1, \ldots, \alpha_k$ are any real numbers.

In [2], Ando also mentioned the relation between the geometric mean and the inverse of matrices for two positive definite matrices; namely,

$$G(M_1^{-1}, M_2^{-1}) = G^{-1}(M_1, M_2).$$

In fact, it can be extended to an identity for α-power mean of two positive matrices.

$$\begin{aligned}
& G^{(\alpha)}\left(M_1^{-1}, M_2^{-1}\right) \\
= & \left(M_2^{-1}\right)^{\frac{1}{2}}\left(\left(M_2^{-1}\right)^{-\frac{1}{2}} M_1^{-1}\left(M_2^{-1}\right)^{-\frac{1}{2}}\right)^{\alpha}\left(M_2^{-1}\right)^{\frac{1}{2}} \\
= & \left(M_2^{\frac{1}{2}}\right)^{-1}\left(\left(M_2^{-\frac{1}{2}} M_1 M_2^{-\frac{1}{2}}\right)^{-1}\right)^{\alpha}\left(M_2^{\frac{1}{2}}\right)^{-1} \\
= & \left(M_2^{\frac{1}{2}}\left(M_2^{-\frac{1}{2}} M_1 M_2^{-\frac{1}{2}}\right)^{\alpha} M_2^{\frac{1}{2}}\right)^{-1} \\
= & \left(G^{(\alpha)}(M_1, M_2)\right)^{-1}.
\end{aligned}$$

A quick induction is sufficient to extend this to the general α-power mean of more than two matrices.

Lemma 3. *Let M_i $(1 \leq i \leq k)$ be positive definite $n \times n$ matrices, and let α_i $(1 \leq i \leq k)$ be real scalars. Then*

$$G^{(\alpha_1,\ldots,\alpha_{k-1})}\left(M_1^{-1}, \ldots, M_k^{-1}\right) = \left(G^{(\alpha_1,\ldots,\alpha_{k-1})}(M_1, \ldots, M_k)\right)^{-1}.$$

In particular,

$$G_w\left(M_1^{-1}, \ldots, M_k^{-1}\right) = \left(G_w\left(M_1, \ldots, M_k\right)\right)^{-1}.$$

Using these we can prove the following generalization of the arithmetic-geometric-harmonic mean inequality involving tensor products.

Theorem 4. *Let M_{ij} $(1 \leq i \leq m,\ 1 \leq j \leq k)$ be positive definite $n \times n$ matrices, and let w_i $(1 \leq i \leq m)$ be positive scalars summing to 1. Then*

$$\left(\sum_{i=1}^{m} w_i \prod_{j=1}^{k} \otimes M_{ij}^{-1}\right)^{-1} \leq \prod_{j=1}^{k} \otimes\, G_w(M_{1j}, \ldots, M_{mj}) \leq \sum_{i=1}^{m} w_i \prod_{j=1}^{k} \otimes M_{ij} \qquad (1)$$

and

$$\left(\sum_{i_1,\ldots,i_m} w_{i_1}\cdots w_{i_m}\prod_{j=1}^{k}\otimes M_{i_jj}^{-1}\right)^{-1} \le G_w\left(\prod_{j=1}^{k}\otimes M_{1j},\ldots,\prod_{j=1}^{k}\otimes M_{mj}\right)$$

$$\le \sum_{i_1=1}^{m}\cdots\sum_{i_m=1}^{m} w_{i_1}\cdots w_{i_m}\prod_{j=1}^{k}\otimes M_{i_jj}, \tag{2}$$

where the equalities hold if and only if $\prod_{j=1}^{k}\otimes M_{1j} = \cdots = \prod_{j=1}^{k}\otimes M_{mj}$.

Proof: The right hand side of (1) was proved by Feng and Tonge in [4]. For the left hand side of (1), substituting M_{ij}^{-1} for M_{ij} in the right hand side and using Lemma 3, we obtain

$$\prod_{j=1}^{k}\otimes\ (G_w(M_{1j},\ldots,M_{mj}))^{-1} = \prod_{j=1}^{k}\otimes G_w\left(M_{1j}^{-1},\ldots,M_{mj}^{-1}\right) \le \sum_{i=1}^{m} w_i\prod_{j=1}^{k}\otimes M_{ij}^{-1}.$$

Using Lemma 1 and the above inequality, we have

$$\left(\prod_{j=1}^{k}\otimes G_w(M_{1j},\ldots,M_{mj})\right)^{-1} \le \sum_{i=1}^{m} w_i\prod_{j=1}^{k}\otimes M_{ij}^{-1}.$$

Thus,

$$\left(\sum_{i=1}^{m} w_i\prod_{j=1}^{k}\otimes M_{ij}^{-1}\right)^{-1} \le \prod_{j=1}^{k}\otimes G_w(M_{1j},\ldots,M_{mj}).$$

For the right hand side of (2), we use Lemma 2 and the right hand side of (1) to obtain

$$G_w\left(\prod_{j=1}^{k}\otimes M_{1j},\ldots,\prod_{j=1}^{k}\otimes M_{mj}\right) = \prod_{j=1}^{k}\otimes G_w(M_{1j},\ldots,M_{mj})$$

$$\le \prod_{j=1}^{k}\otimes\left(\sum_{i=1}^{m} w_iM_{ij}\right) = \sum_{i_1=1}^{m}\cdots\sum_{i_m=1}^{m} w_{i_1}\cdots w_{i_m}\prod_{j=1}^{k}\otimes M_{i_jj}.$$

For the left hand side of (2), substituting M_{ij}^{-1} for M_{ij} in right hand side, we get

$$G_w\left(\prod_{j=1}^{k}\otimes M_{1j}^{-1},\ldots,\prod_{j=1}^{k}\otimes M_{mj}^{-1}\right) \le \sum_{i_1=1}^{m}\cdots\sum_{i_m=1}^{m} w_{i_1}\cdots w_{i_m}\prod_{j=1}^{k}\otimes M_{i_jj}^{-1}.$$

Using Lemma 2, Lemma 3, and Lemma 1, in that order, gives us

$$\begin{aligned} G_w\left(\prod_{j=1}^{k}\otimes M_{1j}^{-1},\ldots,\prod_{j=1}^{k}\otimes M_{mj}^{-1}\right) &= \prod_{j=1}^{k}\otimes G_w(M_{1j}^{-1},\ldots,M_{mj}^{-1}) \\ &= \prod_{j=1}^{k}\otimes\left(G_w(M_{1j},\ldots,M_{mj})\right)^{-1} \\ &= \left(\prod_{j=1}^{k}\otimes G_w(M_{1j},\ldots,M_{mj})\right)^{-1} \end{aligned}$$

Finally, using Lemma 2 once again, we know that

$$\left(\prod_{j=1}^{k}\otimes G_w(M_{1j},\ldots,M_{mj})\right)^{-1} = \left(G_w\left(\prod_{j=1}^{k}\otimes M_{1j},\ldots,\prod_{j=1}^{k}\otimes M_{mj}\right)\right)^{-1}.$$

As a result,

$$\left(G_w\left(\prod_{j=1}^{k}\otimes M_{1j},\ldots,\prod_{j=1}^{k}\otimes M_{mj}\right)\right)^{-1} \leq \sum_{i_1=1}^{m}\cdots\sum_{i_m=1}^{m} w_{i_1}\cdots w_{i_m}\prod_{j=1}^{k}\otimes M_{i_j j}^{-1}.$$

Hence,

$$\left(\sum_{i_1=1}^{m}\cdots\sum_{i_m=1}^{m} w_{i_1}\cdots w_{i_m}\prod_{j=1}^{k}\otimes M_{i_j j}^{-1}\right)^{-1} \leq G_w\left(\prod_{j=1}^{k}\otimes M_{1j},\ldots,\prod_{j=1}^{k}\otimes M_{mj}\right).$$

□

Alić, Mond, Pečarić and Volence [1] gave a generalization of Theorem 1 in the negative weight case. The various means are defined just as before, even if there are negative weights.

Theorem 5 (Alić, Mond, Pečarić and Volence, [1])**.** *Let* M_i $(1 \leq i \leq k)$ *be positive definite* $n \times n$ *matrices, Let* w_i $(1 \leq i \leq k)$ *be real numbers such that* $w_1 > 0$, $w_i < 0$ $(2 \leq i \leq k)$, *and* $\sum_{l=1}^{k} w_l = 1$. *Then*

$$A_w(M_1,\ldots,M_k) \leq G_w(M_1,\ldots,M_k).$$

If $w_1M_1^{-1} + \cdots + w_kM_k^{-1} > 0$, *then*

$$G_w(M_1,\ldots,M_k) \leq H_w(M_1,\cdots,M_k).$$

Equalities hold if and only if $M_1 = \cdots = M_k$.

From Theorem 5, using a proof similar to Theorem 4, we obtain the following.

Theorem 6. *Let* M_{ij} $(1 \leq i \leq m,\ 1 \leq j \leq k)$ *be positive definite* $n \times n$ *matrices,and let* w_i $(1 \leq i \leq k)$ *be real numbers such that* $w_1 > 0,\ w_i < 0$ $(2 \leq i \leq k)$ *and* $\sum_{l=1}^{k} w_l = 1$*. Then*

$$\prod_{j=1}^{k} \otimes\, G_w(M_{1j}, \ldots, M_{mj}) \geq \sum_{i=1}^{m} w_i \prod_{j=1}^{k} \otimes M_{ij}$$

and

$$G_w\left(\prod_{j=1}^{k} \otimes M_{1j}, \ldots, \prod_{j=1}^{k} \otimes M_{mj}\right) \geq \sum_{i_1=1}^{m} \cdots \sum_{i_m=1}^{m} w_{i_1} \cdots w_{i_m} \prod_{j=1}^{k} \otimes M_{i_j j}.$$

If $w_1 \prod_{j=1}^{k} \otimes M_{1j}^{-1} + \cdots + w_m \prod_{j=1}^{k} \otimes M_{mj}^{-1} > 0$*, then*

$$\left(\sum_{i=1}^{m} w_i \prod_{j=1}^{k} \otimes M_{ij}^{-1}\right)^{-1} \geq \prod_{j=1}^{k} \otimes\, G_w(M_{1j}, \ldots, M_{mj})$$

and

$$\left(\sum_{i_1, \ldots, i_m} w_{i_1} \cdots w_{i_m} \prod_{j=1}^{k} \otimes\, M_{i_j j}^{-1}\right)^{-1} \geq G_w\left(\prod_{j=1}^{k} \otimes M_{1j}, \ldots, \prod_{j=1}^{k} \otimes M_{mj}\right).$$

Equalities hold if and only if $\prod_{j=1}^{k} \otimes M_{1j} = \cdots = \prod_{j=1}^{k} \otimes M_{mj}$.

In some situations, tensor product results can be transferred to results on Hadamard products. This is a consequence of the fact that there is a positive linear map Φ_k from n^k-dimensional Hilbert space to n-dimensional Hilbert space such that, for all $n \times n$ matrices $M_i (1 \leq i \leq k)$,

$$\Phi_k\left(\prod_{i=1}^{k} \otimes M_i\right) = \prod_{i=1}^{k} \circ M_i. \qquad (*)$$

Now using $(*)$, Theorem 4, and Lemma 2, we have the following result for the Hadamard products.

Theorem 7. *Let* M_{ij} $(1 \leq i \leq m,\ 1 \leq j \leq k)$ *be positive definite* $n \times n$ *matrices, and let* w_i $(1 \leq i \leq k)$ *be positive scalars summing to 1. Then*

$$\prod_{j=1}^{k} \circ\, G_w(M_{1j}, \ldots, M_{mj}) \leq \sum_{i_1=1}^{m} \cdots \sum_{i_m=1}^{m} w_{i_1} \cdots w_{i_m} \prod_{j=1}^{k} \circ M_{i_j j}.$$

Using $(*)$ and Theorem 6, we obtain the second result for Hadamard products.

Theorem 8. *Let* M_{ij} $(1 \leq i \leq m,\ 1 \leq j \leq k)$ *be positive definite* $n \times n$ *matrices,and let* w_i $(1 \leq i \leq k)$ *be real numbers such that* $w_1 > 0,\ w_i < 0\ (2 \leq i \leq k)$ *and* $\sum_{l=1}^{k} w_l = 1$*. Then*

$$\prod_{j=1}^{k} \circ\, G_w(M_{1j}, \ldots, M_{mj}) \geq \sum_{i=1}^{m} w_i \prod_{j=1}^{k} \circ M_{ij}.$$

References

[1] M. Alić, B. Mond, J. Pečarić and V. Volence, The Arithmetic-Geometric-Harmonic mean and related matrix inequalities. Linear Algebra Appl. **264**(1997) 55-62.

[2] T. Ando, Concavity of certain maps on positive definite matrices and applications to Hadamard products. Linear Algebra Appl., **26**(1979) 203-241.

[3] Ando, T. and Hiai, F., Hölder type inequalities for matrices. Mathematical Inequalities Appl. **1**(1998) 1-30.

[4] B. Q. Feng and A. M. Tonge, Geometric Means and Hadamard Products, submitted.

[5] M. Sagae and K. Tanabe, Upper and lower bounds for the arithmetic-geometric-harmonic means of positive definite matrices. Linear and Multilinear Algebra, **37**(1994) 279-282.